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EXACT RATIONAL SOLUTIONS OF A TRANSCENDENTAL EQUATION

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ABSTRACT. The equation $x^n b^x = c$ is solved using the Lambert W function. A new simplification rule for W is given that allows those cases in which the equation has rational solutions for x to be identified. A related equation studied by Euler, $x^y = y^x$, is also investigated using W.

RÉSUMÉ. On résout l'équation $x^n b^x = c$ en utilisant la fonction W de Lambert. Une nouvelle règle de simplification pour W est donnée qui permet d'identifier et de calculer certains cas pour lesquels l'équation possède des solutions rationnelles x. Nous examinons aussi à l'aide de la fonction W une équation apparentée à la première et qui fut étudiée par Euler, $x^y = y^x$.

1. Introduction. For $n \in \mathbb{Z}$, $b, c \in \mathbb{Q}$ and $b > 0$, the equation

$$x^n b^x = c \tag{1}$$

has, in general, solutions $x \in \mathbb{C}$. The most interesting cases, however, usually have some solutions $x \in \mathbb{Q}$, and a method is needed to find both types of solution. The general solution of (1) is obtained below in terms of the Lambert W function $W_k(z)$, which is defined by [1]

$$W_k(z) e^{W_k(z)} = z \ , \tag{2}$$

where k identifies the branch of this multivalued inverse function. Thus an automatic solution of (1) by computer is possible, except for the difficulty of identifying rational solutions. Consider $x^2 2^x = 72$, which has an infinite number of non-rational solutions, and the solution $x = 3$. In terms of W, this solution is

$$x = \frac{2}{\ln 2} W_0\left(3\sqrt{2}\ln 2\right) \ ,$$

which implies the simplification $W_0\left(3\sqrt{2}\ln 2\right) = (3/2)\ln 2$, but no rule covering this has been previously reported. Such simplification rules must be investigated before equation (1) can be solved in the best way. In addition, new simplifications may prove useful in other applications of the Lambert W function.

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2. Real solutions and rational solutions. In (1), the case $b<1$ can be avoided by the transformation $x \to -x$, so only the case $b>1$ is considered.

Theorem 1. Let $b, c \in \mathbb{R}$, $b>1$ and $n \in \mathbb{Z}$. Let $\sqrt[n]{c}$ denote the real branch of $c^{1/n}$. The solutions $x \in \mathbb{R}$ of the equation $x^n b^x = c$ are as follows.

1. If n is odd and $c > -(n/e\ln b)^n$ or if n is even and $c \geq 0$,

$$x = \frac{n}{\ln b} W_0\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$

2. If n is odd and $0 > c > -(n/e\ln b)^n$,

$$x = \frac{n}{\ln b} W_{-1}\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$

3. If n is even and $0 < c \leq (n/e\ln b)^n$,

$$x = \frac{n}{\ln b} W_0\left(-\frac{\ln b}{n}\sqrt[n]{c}\right) , \quad \text{or} \quad x = \frac{n}{\ln b} W_{-1}\left(-\frac{\ln b}{n}\sqrt[n]{c}\right) .$$

Proof: Consider the case for n odd. Take the real-branch nth root of both sides of (1). Then it becomes, because n is odd,

$$xb^{x/n} = xe^{(x/n)\ln b} = \sqrt[n]{c} ,$$

and multiplying through by $(\ln b)/n$ and comparing with (2) establishes that

$$x = \frac{n}{\ln b} W_k\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$

Only the branches $k=0$ and $k=-1$ of W take real values [1]. The principal branch $W_0(x)$ is real for $x \geq -1/e$, while $W_{-1}(x)$ is real for $-1/e < x < 0$. Imposing these restrictions leads to the cases listed in the theorem. The other cases are similar. It is clear that the converse statements are also true.

Example. The solutions of $x^2(9/16)^x = 3/16$ are

$$\begin{aligned} \frac{-2}{\ln(16/9)} W_0\left(\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) &\approx -.387351650 , \\ \frac{-2}{\ln(16/9)} W_0\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) &= \frac{1}{2} , \\ \frac{-2}{\ln(16/9)} W_{-1}\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) &\approx 11.35508246 . \end{aligned}$$

One must be aware of the possibility of equivalent expressions:

$$\frac{-2}{\ln(16/9)} W_0\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) = \frac{-1}{\ln(4/3)} W_0(-\frac{\sqrt{3}}{4}\ln\frac{4}{3}) = \frac{1}{2} .$$

We are led by the above to seek an algorithm for deciding whether $W_k(r_1 \sqrt[n]{r_2} \ln b)$, with $r_1, r_2, b \in \mathbb{Q}$, simplifies to $r \ln b$, with $r \in \mathbb{Q}$. A standard method for obtaining rational numbers from approximate data is to truncate a continued-fraction expansion [5], and therefore the obvious strategy is to evaluate $W/\ln b$ as a floating point number and then to expand the result as a continued fraction. This idea has two drawbacks, and they are related: first, the number of digits at which the computation is done will determine its success, and second, it is not a decision procedure, because failure to find a rational simplification is no proof that one does not exist. For example, consider the following problem, worked first using 10 digits.

$$W_0\left(\frac{528309}{168976}\ln\left(\frac{9}{4}\right)3^{\frac{7451}{10561}}2^{\frac{3110}{10561}}\right)\Big/\ln\frac{9}{4} \approx 1.852760155\ . \tag{3}$$

The continued-fraction expansion, in the notation of [5], begins

$$1.852760155 \approx [1,1,5,1,3,1,3,1,63,1,30]\ .$$

Re-computed using 16 digits, the continued fraction becomes

$$1.852760155288325 \approx [1,1,5,1,3,1,3,1,64,291347701]\ .$$

Therefore, it is only at 16 digits of precision that we are led to conjecture, correctly, that $r = [1,1,5,1,3,1,3,1,64]$; in other words

$$W_0\left(\frac{528309}{168976}\ln\left(\frac{9}{4}\right)3^{\frac{7451}{10561}}2^{\frac{3110}{10561}}\right) = \frac{19567}{10561}\ln\frac{9}{4}\ .$$

If this method fails, it might be that no rational simplification exists, or it might be merely that the precision used was insufficient. An algorithm that guarantees a decision is based on the following definition and theorem.

Definition. The integer power content of $b \in \mathbb{Q}$ is the largest $m \in \mathbb{Z}$ such that $b = \hat{b}^m$ and $\hat{b} \in \mathbb{Q}$. We write $m = \operatorname{ipc} b$, and call $\hat{b}$ the integer-power-free reduction of b, written $\hat{b} = \operatorname{ipf} b$. Clearly $b = (\operatorname{ipf} b)^{\operatorname{ipc} b}$.

Theorem 2. Given $r_1, r_2, b \in \mathbb{Q}$, the simplification of $W_k(r_1 \sqrt[n]{r_2} \ln b)$ to the form $r \ln b$, with $r \in \mathbb{Q}$, is possible if and only if $k = 0$ or $k = -1$ and

$$x = \frac{n}{\ln(\operatorname{ipf} b)} W_k(r_1 \sqrt[n]{r_2} \ln b) \in \mathbb{Z}\ , \tag{4}$$

in which case $r = x \operatorname{ipc}(b)/n$. Moreover, x is an integral solution of

$$(\operatorname{ipf} b)^x - \left(\frac{r_1 n \operatorname{ipc} b}{x}\right)^n r_2 = 0\ . \tag{5}$$

Proof: By theorem 1, x as defined in (4) is a solution of

$$x^n(\operatorname{ipf} b)^x = c = r_1^n n^n (\operatorname{ipc} b)^n r_2 \ . \tag{6}$$

Then $x \in \mathbb{Q} \Rightarrow x \in \mathbb{Z}$, because if $x = p/q$, with $p, q \in \mathbb{Z}$, then

$$(\operatorname{ipf} b^p)^{1/q} = c(q/p)^n \ .$$

Clearly the right-hand side is rational, but by the construction of $\operatorname{ipf} b$, the left side is not in $\mathbb{Q}$ unless $q = 1$.

Computational algorithm. A practical procedure is as follows. The quantity x defined in (4) is evaluated in two stages. At first, the order of magnitude of x is determined; let it be $O(10^p)$. The evaluation is then repeated using more digits of precision than p, so that if x is indeed an integer, its value will be determined correctly.[1] The proposed integral value is then accepted if it satisfies (5), otherwise there is no rational simplification. If a computer system verifies (5) by direct computation of the left-hand side, very large integers are generated. It would therefore be wise for a system to use the laws of exponents first.

Example. Returning to problem (3), one now computes

$$\frac{10561}{\ln(3/2)} W_0 \left(\frac{528309}{168976} \ln\left(\frac{9}{4}\right) 3^{\frac{7451}{10561}} 2^{\frac{3110}{10561}} \right) \approx 39135.$$

where 5 digits of precision have been used. The size of the result shows that a computation accurate to 6 digits is sufficient (less than half the 16 needed by the continued fraction method), and this gives 39134.0 as the result. We now must verify that

$$(3/2)^{39134} - \left(\frac{528309}{168976} \frac{(10561)2}{39134} \right)^{10561} 3^{7451} 2^{3110} = 0 \ .$$

A brute-force evaluation of the left-hand side takes about 20 seconds on existing computers, but working in powers of 2 and 3 gives rapid verification.

3. A problem solved by Euler. An interesting related equation was studied by Euler and others, namely,

$$x^y = y^x \ .$$

Regarding y as a given quantity makes this equation a special case of (1). For $x > 0$ and $y > 0$ it is straightforward to obtain the solution $x = -yW_k(-\ln y/y)/\ln y$. In addition to the solution $x = y$, the equation has the parametric solutions

$$x = (1+s)^{1/s} \ , \qquad y = (1+s)^{1+1/s} \ ,$$

[1] Arithmetic in Maple V release 5 is guaranteed to produce results accurate to 0.6 ulp for fundamental operations such as multiplication and single function evaluations. Other systems are similar. Evaluation of a simple composition of these operations, such as $aW(b \ln c)$ for numeric a, b and c, can be proved accurate—with some work—if enough guard digits are used in the computation.

and the symmetric solution (y, x). This implies simplification rules for W. We find from the $y = x$ solution, setting $y = 1/t$,

$$\begin{aligned} W_0(t \ln t) &= \ln t \;, \quad \text{for} \quad t > 1/e \;, \\ W_{-1}(t \ln t) &= \ln t \;, \quad \text{for} \quad 0 < t < 1/e \;. \end{aligned}$$

From the parametric solution, we obtain the additional relations

$$\begin{aligned} W_0\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s}\ln(1+s) \quad \text{for} \; -1 < s < 0 \;, \\ W_{-1}\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s}\ln(1+s) \quad \text{for} \; 0 < s \;. \end{aligned}$$

These results are equivalent to a result by Lauwerier [3, 4]. Equations (54–57) in [3] are obtained by writing $s = e^{2\Delta} - 1$. As of this writing, there are no computer algebra implementations of these (domain-specific) simplification rules.

4. The complex case. If all variables are allowed to be complex, then theorem 1 can be generalized, but without the same degree of completeness. If it is assumed that z^α and b^z are the principal complex values of the power function, we give a formula that generates all solutions of $z^\alpha b^z = c$, but in addition generates solutions that correspond to other branches of the power function. Theorem 3 uses the unwinding number $\mathcal{K}$, but defined as the negative of the unwinding number defined in [2]. Thus, it is now defined for $z \in \mathbb{C}$ as

$$\ln e^z = z - 2\pi i \mathcal{K}(z) \;. \tag{7}$$

After working with $\mathcal{K}$, it has become clear to us that this new definition is better, because fewer minus signs arise in the equations.

Theorem 3: Let $\alpha, b, c \in \mathbb{C}$. If $z \in \mathbb{C}$ is a solution of $z^\alpha b^z = c$, then it satisfies

$$z = \frac{\alpha}{\ln b} W_k\left(\frac{\ln b}{\alpha} c^{1/\alpha} \exp\left[\frac{2\pi i}{\alpha}\mathcal{K}(\alpha \ln z + z \ln b)\right]\right) \;, \tag{8}$$

where k is any integer and $\mathcal{K}$ is the unwinding number.

Proof: Take the $1/\alpha$ power of both sides:

$$(z^\alpha b^z)^{1/\alpha} = c^{1/\alpha} \;,$$

where all powers are principal value. In terms of the unwinding number,

$$(z^\alpha b^z)^{1/\alpha} = z b^{z/\alpha} \exp\left(-\frac{2\pi i}{\alpha}\mathcal{K}(\alpha \ln z + z \ln b)\right) \;.$$

The solution of the equation then follows as stated.

The theorem can be used algorithmically to generate solutions for $z^\alpha b^z = c$, by taking advantage of the fact that both k and $\mathcal{K}$ are integers. Stepping through the ranges of k and $\mathcal{K}$ generates all possible solutions, and some spurious solutions. Each new solution must to be verified in the original equation.

Example. For the equation $x^{1/3}256^x = 1$, the exponential factor in (8) is 1, but even so, W_0 and W_3 give solutions, but W_{-1}, W_1, W_2 do not, unless $x^{1/3}$ takes values from its other branches.

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